

# UNIVERSAL SPACES FOR UNRAMIFIED GALOIS COHOMOLOGY

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**ABSTRACT.** We construct and study universal spaces for birational invariants of algebraic varieties over algebraic closures of finite fields.

## INTRODUCTION

Let  $\ell$  be a prime. Recall that in topology, there exist unique (up to homotopy) topological spaces  $K(\mathbb{Z}/\ell^n, m)$  such that

- $K(\mathbb{Z}/\ell^n, m)$  is homotopically trivial up to dimension  $m - 1$ , in particular,

$$H^i(K(\mathbb{Z}/\ell^n, m), \mathbb{Z}/\ell^n) = 0, \quad \text{for } i < m;$$

- $H^m(K(\mathbb{Z}/\ell^n, m), \mathbb{Z}/\ell^n)$  is cyclic, with a distinguished generator  $\kappa_m$ ;
- for every topological space  $X$  and every  $\alpha \in H^m(X, \mathbb{Z}/\ell^n)$  there is a unique, up to homotopy, continuous map

$$\mu_{X,\alpha} : X \rightarrow K(\mathbb{Z}/\ell^n, m)$$

such that

$$\mu_{X,\alpha}^*(\kappa_m) = \alpha.$$

This reduces many questions about singular cohomology to the study of these universal spaces. Analogous theories exist for other contravariant functors, for example, topological K-theory, or the theory of cobordisms. The study of moduli spaces in algebraic geometry can be viewed, broadly speaking, as an incarnation of the same idea of universal spaces.

Here we propose a similar theory for unramified cohomology, developed in connection with the study of birational properties of algebraic varieties [4], [16]. The Bloch–Kato conjecture proved by Voevodski, Rost, and Weibel, combined with techniques and results from birational anabelian geometry in [10], implies that an unramified class in the cohomology of a function field  $K = k(X)$  of an algebraic variety  $X$  over an algebraic closure of a finite field  $k = \bar{\mathbb{F}}_p$ , with finite constant

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coefficients, is induced from the cohomology of a finite abelian group  $G^a$ . Our main result is:

**Theorem.** *Let  $\ell$  and  $p$  be distinct primes,  $K = k(X)$  the function field of an algebraic variety  $X$  of dimension  $\geq 2$  over  $k = \mathbb{F}_p$ ,  $G_K$  its absolute Galois group, and  $\alpha_K \in H_{nr}^*(G_K, \mathbb{Z}/\ell^n)$  an unramified class. Then there exists a finite set  $J$  of finite-dimensional  $k$ -vector spaces  $V_j, j \in J$ , such that  $\alpha_K$  is induced, via a rational map, from an unramified class in the cohomology of an explicit open subset of the quotient of*

$$\mathbb{P} := \prod_{j \in J} \mathbb{P}(V_j)$$

*by a finite abelian  $\ell$ -group  $G^a$ , acting projectively on each factor.*

Thus, the spaces  $\mathbb{P}/G^a$  serve as universal spaces for *all* finite birational invariants of algebraic varieties over  $k = \mathbb{F}_p$ .

Actions of finite abelian groups  $G^a$  on products of projective spaces are described by central extensions of  $G^a$ , i.e., by subspaces in  $\wedge^2(G^a)$ . This allows to present unramified classes of  $X$  in terms of configurations of subspaces of skew-symmetric matrices. For example, if the unramified Brauer group of  $X$  is trivial, then all finite birational invariants of  $X$  are encoded already in the combinatorics of configurations of liftable subgroups in finite abelian quotients of the absolute Galois group  $G_K$  (see Section 1 for the definition).

Similar ideas were first put forward in [4] and [5], however, the recent proof of the Bloch–Kato conjecture allows us to formulate a more precise and constructive theory. This approach to birational invariants leads to many new questions:

- Is there a smaller class of configurations with this universal property?
- How does this structure interact with Sylow subgroups of  $G_K$ ?
- Is there an extension to cohomology with  $\mathbb{Z}_\ell$ -coefficients? An equally simple description of models for  $\ell$ -adic invariants would provide insights into higher-dimensional Langlands correspondence.
- What are the analogs of universal spaces for varieties over  $k = \bar{\mathbb{Q}}$ ? Counterexamples to our main result arise from bad reduction places, already for abelian varieties [4].

Here is the roadmap of the paper: In Section 1 we recall basic facts about stable and unramified cohomology. In Section 3 we provide some background on valuation theory. In Section 5 we investigate Galois cohomology groups of function fields of higher-dimensional algebraic varieties over  $k = \mathbb{F}_p$  and their images in cohomology of finite groups.

In Section 6 we introduce and study unramified cohomology of algebraic varieties. Section 7 contains the proof of our main theorem, modulo geometric considerations presented in Sections 8 and 9.

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## 1. STABLE COHOMOLOGY

Let  $G$  be a pro-finite group. We will write

$$G^a = G/[G, G] \quad \text{and} \quad G^c = G/[[G, G], G]$$

for the abelianization, respectively, the second lower central series quotient of  $G$ . We have a canonical central extension

$$(1.1) \quad 1 \rightarrow Z \rightarrow G^c \xrightarrow{\pi_a} G^a \rightarrow 1.$$

Let  $M$  be a topological  $G$ -module and  $H^i(G, M)$  its (continuous)  $i$ -cohomology group. These groups are contravariant with respect to  $G$  and covariant with respect to  $M$ . In this paper,  $G$  is either a finite group or a Galois group (see [1] for background on group cohomology and [28] for background on Galois cohomology) and  $M$  either  $\mathbb{Z}/\ell^n$  or  $\mathbb{Q}/\mathbb{Z}$ , with trivial  $G$ -action. We will sometimes omit the coefficient module  $M$  from the notation.

Our goal is to investigate incarnations of Galois cohomology of function fields in cohomology of finite groups. For example, let  $K = k(X)$  be a function field of an algebraic variety  $X$  over an algebraically closed field  $k$ ; varieties birational to  $X$  are called *models* of  $K$ . Let  $G_K$  be the absolute Galois group of  $K$  and  $\hat{\pi}_1(X)$  the étale fundamental group of  $X$ , with respect to some basepoint. The choice of a base point will not affect our considerations and we omit it from our notation. We have natural homomorphisms

$$H^*(\hat{\pi}_1(X)) \xrightarrow{\kappa_X^*} H_{et}^*(X) \xrightarrow{\tilde{\eta}_X^*} H^*(G_K),$$

where the right arrow arises from the embedding of the generic point  $X_\eta \rightarrow X$ . We will write

$$\eta_X : G_K \rightarrow \hat{\pi}_1(X)$$

and

$$\eta_X^* = \tilde{\eta}_X^* \circ \kappa_X^* : H^*(\hat{\pi}_1(X)) \rightarrow H^*(G_K)$$

for the corresponding map in cohomology.

We say that a class  $\alpha_K \in H^*(G_K)$  is defined (or represented) on a model  $X$  of  $K$  if there exists a class  $\alpha_X \in H_{et}^*(X)$  such that

$$\alpha_K = \tilde{\eta}_X^*(\alpha_X).$$

Let  $G$  be a finite group. A continuous homomorphism

$$\chi : \hat{\pi}_1(X) \rightarrow G$$

gives rise to homomorphisms in cohomology

$$H^*(G) \xrightarrow{\chi^*} H^*(\hat{\pi}_1(X)) \xrightarrow{\eta_X^*} H^*(G_K).$$

Conversely, every  $\alpha_K \in H^*(G_K)$  arises in this way: there exist

- a model  $X$  of  $K$ ,
- a continuous homomorphism  $\chi$  as above,
- and a class  $\alpha_G \in H^*(G)$

such that

$$\alpha_K = \eta_X^*(\chi^*(\alpha_G)).$$

This follows from the description of étale cohomology of points, see [22]. In such situations we say that  $\alpha_K$  is defined on  $X$  and is induced from  $\chi$ .

A version of this construction arises as follows: assume that the characteristic of  $k$  does not divide the order of  $G$ . Let  $V$  be a faithful representation of  $G$  over  $k$ , and  $X$  an algebraic variety over  $k$  with function field

$$K = k(X) \simeq k(V)^G,$$

the field of invariants; we will write  $X = V/G$  and call it a *quotient*. Even more generally, let  $Y$  be an algebraic variety over  $k$  with a generically free action of  $G$ ,  $X = Y/G$  a quotient. This situation gives rise to a natural surjective continuous homomorphism

$$G_K \rightarrow G$$

and induced homomorphisms on cohomology

$$s_K^i : H^i(G) \rightarrow H^i(G_K).$$

The following lemma shows that we have many choices in realizing a class  $\alpha_K \in H^i(G_K)$ :

**Lemma 1.1.** [4] *Assume that  $\alpha_K \in H^i(G_K)$  is represented by a class  $\alpha_X \in H_{et}^i(X)$  on some affine irreducible model  $X$  of  $K$  and is induced from a surjective continuous homomorphism  $\chi : \hat{\pi}_1(X) \rightarrow G$  and a class  $\alpha_G \in H^i(G)$ . Let  $V$  be a faithful representation of  $G$  over  $k$  and*

$V^\circ \subset V$  the locus where the action is free. Then, for every  $x \in X$  and  $v \in V^\circ$  there exists a map

$$f = f_x : X \rightarrow V/G$$

such that

- $f(x) = v$  and
- the restriction of  $\alpha_X$  to  $X^\circ = f^{-1}(V^\circ/G) \subset X$  is equal to  $f^*(\alpha_G)$ .

*Proof.* We follow the proof in [4]. The homomorphism  $\chi : \hat{\pi}_1(X) \rightarrow G$  defines a finite étale covering  $\pi : \tilde{X} \rightarrow X$ , by an affine variety  $\tilde{X}$ . The ring  $k[\tilde{X}]$  is a  $k[G]$ -module. Every finite-dimensional  $k[G]$ -submodule  $W^* \subset k[\tilde{X}]$  defines a  $G$ -equivariant map  $\tilde{X} \rightarrow W := \text{Spec}(W^*)$ .

Let  $e \in k[G]$  be the unit element of  $G$ . For any  $G$ -orbit  $G \cdot y \in V$  there is a  $G$ -linear homomorphism

$$l_y : k[G] \rightarrow V,$$

which maps the orbit  $G \cdot e$  to  $G \cdot y$ . Let  $\tilde{x} \in \pi^{-1}(x)$ . Choose  $h \in k[\tilde{X}]$  such that

$$h(\tilde{x}) = 1, \quad h(g \cdot \tilde{x}) = 0, \quad g \neq e.$$

Then  $h$  generates a  $k[G]$ -submodule  $W \subset k[\tilde{X}]$  and defines a regular  $G$ -map  $h : \tilde{X} \rightarrow W = k[G]$ , with  $h(\tilde{x}) = e \in k[G]$ . The map  $f := l_y \circ h$  is a regular  $G$ -map satisfying the first property.

Let  $X^\circ = f^{-1}(V^\circ/G) \subset X$ . It is a nonempty affine subvariety. We have a compatible diagram of  $G$ -maps

$$\begin{array}{ccc} V^\circ & \longleftarrow & \tilde{X}^\circ \\ \pi_G \downarrow & & \downarrow \pi_0 \\ V^\circ/G & \xleftarrow{f} & X^\circ \end{array} \quad \subset \quad \begin{array}{ccc} \tilde{X} & & \\ & \downarrow \pi & \\ X & & \end{array}$$

and the maps  $\pi_G$  and  $\pi$  induce the same cover  $\pi_0$ . This implies the second claim.  $\square$

We can achieve even more flexibility for  $G$ -maps, under a *projectivity* conditions on  $V$ : we say that a  $G$ -module  $V$  is projective if for every finite-dimensional representation  $W$  of  $G$  with a  $G$ -surjection

$$\mu : W \rightarrow V$$

there exists a  $G$ -section

$$\theta : V \rightarrow W \quad \text{with} \quad \mu \circ \theta = \text{id}.$$

This condition holds, for example, for regular representations over arbitrary fields or when the order of  $G$  is coprime to the characteristic of the ground field  $k$ .

Now let  $\{S_j\}_{j \in J}$  be a finite set of  $G$ -orbits in  $Y$  with stabilizers  $H_j$  so that  $S_j \simeq G/H_j$ . Consider a faithful representation  $V$  of  $G$  and a subset  $\{T_j\}_{j \in J}$  of  $G$ -orbits in  $V$  with stabilizers  $Q_j$ , with  $H_j \subset Q_j$ ,  $T_j = G/Q_j$ . Consider regular  $G$ -maps  $f_j : S_j \rightarrow T_j$ , for  $j \in J$ .

**Lemma 1.2.** *Assume that  $V$  is a projective  $G$ -module. Then there is a regular  $G$ -map  $f : Y \rightarrow V$  such that  $f = f_j$ , for all  $j \in J$ .*

We return to our setup:  $X = Y/G$ ,  $K = k(X)$ , and  $\chi : G_K \rightarrow G$ , inducing

$$s_K^i : H^*(G) \rightarrow H^*(G_K).$$

The groups

$$H_{s,K}^i(G) := H^i(G)/\text{Ker}(s_K^i)$$

are called *stable cohomology groups* with respect to  $K = k(X)$ . Let

$$\text{Ker}(s^i) := \bigcap_K \text{Ker}(s_K^i),$$

over all function fields  $K = k(X)$  as above. In fact,

$$\text{Ker}(s^i) = \text{Ker}(s_{k(V/G)}^i),$$

for some faithful representation  $V$  of  $G$  over  $k$ , in particular, this is independent of the choice of  $V$  (see [8, Proposition 4.3]). The groups

$$H_s^i(G) := H^i(G)/\text{Ker}(s^i)$$

are called *stable cohomology groups* of  $G$  (with coefficients in  $M = \mathbb{Z}/\ell^n$  or  $\mathbb{Q}/\mathbb{Z}$ ); they depend on the ground field  $k$ . These define contravariant functors in  $G$ . For example, for a subgroup  $H \subset G$  we have a restriction homomorphism

$$\text{res}_{G/H} : H_s^*(G) \rightarrow H_s^*(H).$$

Furthermore:

- While usual group cohomology  $H^i(G)$  can be nontrivial for infinitely many  $i$  (even for cyclic groups), *stable* cohomology groups  $H_s^i(G) = 0$  for  $i > \dim(V)$ , where  $V$  is a faithful representation.
- We have

$$H_s^i(G) \subseteq H_s^i(\text{Syl}_\ell(G))^{N_G(\text{Syl}_\ell(G))},$$

where  $N_G(H)$  is the normalizer of  $H$  in  $G$  and  $\text{Syl}_\ell(G)$  an  $\ell$ -Sylow subgroup of  $G$ .

The determination of the stable cohomology ring

$$H_s^*(G) := \bigoplus_i H_s^i(G)$$

is a nontrivial problem, see, e.g., [7] for a computation of stable cohomology of alternating groups. For finite abelian groups  $G$ , we have

$$(1.2) \quad H_s^*(G) = \wedge^*(H^1(G)).$$

For central extensions of finite groups as in (1.1), the kernel of

$$\pi_a^* : H_s^*(G^a) \rightarrow H_s^*(G^c)$$

is the ideal  $I = I(G^c)$  generated by

$$R^2 = R^2(G^c) := \text{Ker} (H_s^2(G^a) \rightarrow H_s^2(G^c)).$$

(see, for example, [12, Section 8]). We will identify

$$H_s^*(G^a)/I$$

with its image in  $H_s^*(G^c)$ . An important role in the computation of this subring of  $H_s^*(G^c)$  is played by the *fan*

$$\Sigma = \Sigma(G^c) = \{\sigma\},$$

the set of *noncyclic* liftable subgroups  $\sigma$  of  $G^a$ , and the *complete fan*

$$\bar{\Sigma} = \bar{\Sigma}(G^c) = \{\sigma\},$$

consisting of *all* liftable subgroups  $\sigma \subset G^c$ : a subgroup  $\sigma$  is *liftable* if and only if the full preimage  $\tilde{\sigma}$  of  $\sigma$  in  $G^c$  is abelian. The fan  $\Sigma$  defines a subgroup  $R^2(\Sigma) \subseteq H_s^2(G^a)$  as the set of all elements which vanish upon restriction to every  $\sigma \in \Sigma$ . We have

$$R^2 \subseteq R^2(\Sigma).$$

**Lemma 1.3.** *For every  $\alpha \in I(G^c) \subseteq H_s^*(G^a)$  and every  $\sigma \in \Sigma(G^c)$  the restriction of  $\alpha$  to  $\sigma$  is trivial.*

**Definition 1.4.** Let

$$1 \rightarrow Z \rightarrow G^c \rightarrow G^a \rightarrow 1$$

be a central extension. A  $\Delta$ -pair  $(I, D)$  of  $G^a$  is a set of subgroups

$$I \subseteq D \subseteq G^a$$

such that

- $I \in \bar{\Sigma}(G^c)$ ,
- $D$  is noncyclic,
- for every  $\delta \in D$ , the subgroup  $\langle I, \delta \rangle \in \bar{\Sigma}(G^c)$ .

This definition depends on  $G^c$ . Assume we have a commutative diagram of central extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{G}^c & \longrightarrow & \tilde{G}^a \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \gamma \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a \longrightarrow 1. \end{array}$$

**Definition 1.5.** A  $\Delta$ -pair  $(\tilde{I}, \tilde{D})$  of  $\tilde{G}^a$  *surjects* onto a  $\Delta$ -pair  $(I, D)$  of  $G^a$  if  $\gamma(\tilde{I}) = I$  and  $\gamma(\tilde{D}) = D$ .

**Definition 1.6.** A class  $\alpha \in H_s^i(G^a)$  is *unramified* with respect to a  $\Delta$ -pair if its restriction to  $D$  is induced from  $D/I$ , i.e., there exists a  $\beta \in H_s^i(D/I)$  such that  $\phi(\alpha) = \psi(\beta)$ , for the natural homomorphisms in the diagram:

$$H_s^*(G^a) \xrightarrow{\phi} H_s^*(D) \xleftarrow{\psi} H_s^*(D/I)$$

By Lemma 1.3 we have a similar notion for

$$H_s^*(G^a)/I(G^c) \subseteq H_s^*(G^c).$$

**Lemma 1.7.** *Consider a homomorphism*

$$\gamma : \tilde{G}^a \rightarrow G^a$$

*and a class  $\alpha \in H^i(G^a)$ . Let  $\tilde{\alpha} := \gamma^*(\alpha) \in H^i(\tilde{G}^a)$  be the induced class. Let  $(\tilde{I}, \tilde{D})$  be a  $\Delta$ -pair in  $\tilde{G}^a$ . Assume that one of the following holds:*

- $\gamma(\tilde{I}) = 0$ ,
- $\gamma(\tilde{D})$  is cyclic,
- $\gamma$  induces a surjection of  $\Delta$ -pairs

$$(\tilde{I}, \tilde{D}) \rightarrow (I, D)$$

*and  $\alpha \in H_s^i(G^a)$  is unramified with respect to  $(I, D)$ .*

*Then  $\tilde{\alpha} \in H_s^i(\tilde{G}^a)$  is unramified with respect to  $(\tilde{I}, \tilde{D})$ .*

*Proof.* The first two cases are evident. Consider the third condition. By assumption,  $\gamma$  induces a homomorphism  $\tilde{D}/\tilde{I} \rightarrow D/I$ . Passing to cohomology we get a commuting diagram

$$\begin{array}{ccc} H_s^*(D/I) & \longrightarrow & H_s^*(D) \\ \downarrow & & \downarrow \\ H_s^*(\tilde{D}/\tilde{I}) & \longrightarrow & H_s^*(\tilde{D}), \end{array}$$

and thus the claim. □



## 2. CENTRAL EXTENSIONS AND ISOCLINISM

Let  $G^a$  and  $Z$  be finite abelian  $\ell$ -groups. Central extensions of  $G^a$  by  $Z$  are parametrized by  $H^2(G^a, Z)$ ; for  $\alpha \in H^2(G^a, Z)$  we let  $G_\alpha^c$  be the corresponding central extension:

$$(2.1) \quad 1 \rightarrow Z \rightarrow G_\alpha^c \xrightarrow{\pi_a} G^a \rightarrow 1$$

Fix an embedding  $Z \hookrightarrow (\mathbb{Q}/\mathbb{Z})^r$ , consider the exact sequence

$$1 \rightarrow Z \rightarrow (\mathbb{Q}/\mathbb{Z})^r \rightarrow (\mathbb{Q}/\mathbb{Z})^r \rightarrow 1,$$

and the induced long exact sequence in cohomology

$$H^1(G^a, (\mathbb{Q}/\mathbb{Z})^r) \xrightarrow{\delta} H^2(G^a, Z) \rightarrow H^2(G^a, (\mathbb{Q}/\mathbb{Z})^r).$$

We say that  $\alpha, \tilde{\alpha} \in H^2(G^a, Z)$  and the corresponding extensions are *isoclinic* if

$$\alpha - \tilde{\alpha} \in \delta(H^1(G^a, (\mathbb{Q}/\mathbb{Z})^r)).$$

This notion does not depend on the chosen embedding  $Z \hookrightarrow (\mathbb{Q}/\mathbb{Z})^r$  and coincides with the standard definition of isoclinic in the theory of  $\ell$ -groups.

**Lemma 2.1.** *If  $\alpha, \tilde{\alpha} \in H^2(G^a, Z)$  are isoclinic then the corresponding extensions of  $G^a$  define the same set of  $\Delta$ -pairs in  $G^a$ .*

*Proof.* A pair of subgroups  $(I, D)$  is a  $\Delta$ -pair in  $G^a$ , with respect to a central extension  $G^c$ , if their preimages commute in  $G^c$ , i.e.,

$$[\pi_a^{-1}(I), \pi_a^{-1}(D)] = 0 \quad \text{in} \quad Z.$$

Consider the homomorphism

$$\pi_a^* : H^2(G^a, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G^c, \mathbb{Q}/\mathbb{Z}),$$

and note that  $\text{Ker}(\pi_a^*)$  only depends on the isoclinism class of the extension. Furthermore,  $H^2(G^a, \mathbb{Q}/\mathbb{Z})$  is dual to  $\wedge^2(G^a)$ . Let  $R \subset \wedge^2(G^a)$  be the subgroup which is dual to  $\text{Ker}(\pi_a^*)$ . It remains to observe that  $(I, D)$  is a  $\Delta$ -pair for  $G^c$  if and only if  $\pi_a^{-1}(I) \wedge \pi_a^{-1}(D)$  intersects  $R$  trivially; thus the notion of a  $\Delta$ -pair is an invariant of the isoclinism class of the extension.  $\square$

**Lemma 2.2.** *If  $\alpha, \tilde{\alpha} \in H^2(G^a, Z)$  are isoclinic then*

$$R^2(G_\alpha^c) = R^2(G_{\tilde{\alpha}}^c),$$

*in particular*

$$H_s^*(G^a, \mathbb{Z}/\ell^n)/I(G_\alpha^c) = H_s^*(G^a, \mathbb{Z}/\ell^n)/I(G_{\tilde{\alpha}}^c).$$

*Proof.* See Section 2 and Theorem 3.2 in [6].  $\square$

**Lemma 2.3.** *If  $R^2 = R^2(\Sigma)$  then  $\Sigma$  determines  $G^c$  up to isoclinism.*

*Example 2.4.* Consider  $G^a = \mathbb{F}_\ell^{2m}$ , as a vector-space over  $\mathbb{F}_\ell$  of dimension  $m$ . Let  $\Sigma$  be the set of  $\mathbb{F}_\ell$ -linear subspaces of the form  $\mathbb{F}_\ell^2$ . Then

$$H_s^i(G^c, \mathbb{Z}/\ell^n) = 0, \quad \text{for } i > 2.$$

**Lemma 2.5.** *If  $\alpha, \tilde{\alpha} \in H^2(G^a, \mathbb{Z})$  are isoclinic then there exist faithful representations  $V, \tilde{V}$  of  $G_\alpha^c$  and  $G_{\tilde{\alpha}}^c$  over  $k$  such that  $V/G_\alpha^c$  and  $\tilde{V}/G_{\tilde{\alpha}}^c$  are birational.*

*Proof.* Explicit construction: Let  $\chi_1, \dots, \chi_r$  be a basis of  $\text{Hom}(Z, k^\times)$  and put

$$V := \bigoplus_{j=1}^r V_j \quad \text{and} \quad \tilde{V} = \bigoplus_{j=1}^r \tilde{V}_j,$$

where

$$V_j = \text{Ind}_Z^{G_\alpha^c}(\chi_j) \quad \text{and} \quad \tilde{V}_j = \text{Ind}_Z^{G_{\tilde{\alpha}}^c}(\chi_j).$$

Note that the *projectivizations*  $\mathbb{P}(V_j)$  and  $\mathbb{P}(\tilde{V}_j)$  are canonically isomorphic as  $G^a$ -representations. The group  $(k^\times)^r$  acts on  $V$  and  $\tilde{V}$ , and both  $V/G_\alpha^c$  and  $\tilde{V}/G_{\tilde{\alpha}}^c$  are birational to

$$\left( \prod_{j=1}^r \mathbb{P}(V_j) \right) / G^a \times \left( \prod_{j=1}^r k^\times / \chi_j(Z) \right).$$

□

**Lemma 2.6.** *Consider a central extension of finite  $\ell$ -groups*

$$(2.2) \quad 1 \rightarrow Z \rightarrow G^c \xrightarrow{\pi_a} G^a \rightarrow 1$$

*and fix a primitive  $g \in G^a$ . Then there exists an isoclinic extension*

$$1 \rightarrow \tilde{Z} \rightarrow \tilde{G}^c \xrightarrow{\tilde{\pi}_a} G^a \rightarrow 1$$

*and a lift  $\tilde{g}^c \in \tilde{G}^c$  of  $g$  such that the order of  $\tilde{g}^c$  in  $\tilde{G}^c$  equals the order of  $g \in G^a$ .*

*Proof.* Since  $H^2(\mathbb{Z}/\ell^n, (\mathbb{Q}/\mathbb{Z})^r) = 0$  we can modify the sequence (2.2) by an element in  $\delta(H^1(G^a, (\mathbb{Q}/\mathbb{Z})^r))$  to split the induced extension

$$1 \rightarrow Z \rightarrow Z_g \rightarrow \langle g \rangle \rightarrow 1, \quad Z_g := (\pi_a)^{-1}(\langle g \rangle),$$

this provides a lift of  $g$  of the same order. □

**Corollary 2.7.** *If for some  $g \in Z$  the restriction of  $\pi_a$  to  $\langle g \rangle$  defines an embedding  $\pi_a : \langle g \rangle \hookrightarrow G^a$  then  $G^c$  is isoclinic to  $G^c / \langle g \rangle \times \langle g \rangle$ .*

*Proof.* Consider the central extension

$$(2.3) \quad 1 \rightarrow \langle g \rangle \rightarrow G^c \rightarrow G^c / \langle g \rangle \rightarrow 1.$$

It is defined by an element  $\alpha_g \in H^2(G^c / \langle g \rangle, \langle g \rangle)$ . Consider the embedding  $\langle g \rangle \subset \mathbb{Q}/\mathbb{Z}$ . Then the image of  $\alpha_g$  in  $H^2(G^c / \langle g \rangle, \mathbb{Q}/\mathbb{Z})$  is 0.

Indeed, the extension is induced from a central extensions of abelian quotient groups

$$1 \rightarrow \pi_a(\langle g \rangle) \rightarrow G^a \rightarrow G^a/\pi_a(\langle g \rangle) \rightarrow 1,$$

which is isoclinic to the trivial central extension; the same holds for the exact sequence (2.3).  $\square$

**Lemma 2.8.** *Consider a central extension of finite  $\ell$ -groups*

$$1 \rightarrow Z \rightarrow G^c \rightarrow G^a \rightarrow 1,$$

*let  $\alpha \in H^2(G^a, Z)$  be the corresponding class, and assume that there exists a  $g \in G^a$  lifting to a  $g^c \in G^c$  of the same order. Let  $Z_g \subset G^c$  be the centralizer of  $g^c$ . Then the central extension*

$$0 \rightarrow \langle g^c \rangle \rightarrow Z_g \rightarrow Z_g/\langle g^c \rangle \rightarrow 0$$

*is induced from the extension*

$$0 \rightarrow \langle g \rangle \rightarrow G^a \rightarrow G^a/\langle g \rangle \rightarrow 0.$$

*If  $V$  is a faithful representation of  $G^c$  as in the proof of Lemma 2.5 then  $V/Z_g$  is a vector bundle over  $\tilde{V}/(Z_g/\langle g^c \rangle)$ , for some faithful representation  $\tilde{V}$  of  $Z_g/\langle g^c \rangle$ . Furthermore, if  $K := k(V/Z_g)$  and*

$$s_K^2 : H^2(G^a, \mathbb{Z}/\ell^n) \rightarrow H^2(G_K, \mathbb{Z}/\ell^n)$$

*is the corresponding homomorphism, then  $s_K^2(\alpha)$  is induced from  $G^a/\langle g \rangle$ .*

*Proof.* Immediate from the definitions.  $\square$

**Lemma 2.9.** *Consider a central extension of finite groups*

$$1 \rightarrow Z \rightarrow G^c \xrightarrow{\pi_a} G^a \rightarrow 1$$

*and let  $V = \oplus_j V_j$  be a faithful representation of  $G^c$  as in Lemma 2.5, i.e., each  $V_j = \text{Ind}_Z^{G^c}(\chi_j)$ , where  $\{\chi_j\}_{j \in J}$  is a basis of  $\text{Hom}(Z, k^\times)$ . Let  $\mathbb{P} := \prod_{j \in J} \mathbb{P}(V_j)$ . Then:*

- (1)  $G^a$  acts faithfully on  $\mathbb{P}$ .
- (2) For any subgroup  $\sigma \subset G^a$  the subset of  $\sigma$ -fixed points  $\mathbb{P}^\sigma \subset \mathbb{P}$  is nonempty if and only if  $\sigma \in \bar{\Sigma}(G^c)$ .
- (3) Each irreducible component of  $\mathbb{P}^\sigma$  is a product of projective subspaces of  $\mathbb{P}(V_j)$ , corresponding to different eigenspaces of  $\sigma$  in  $V_j$ , and distinct irreducible components are disjoint.
- (4) Each irreducible component of  $\mathbb{P}^\sigma$  is stable under the action of  $H_\sigma \subset G^c$ , the maximal subgroup such that  $[H_\sigma, \pi_a^{-1}(\sigma)] = 0$  in  $G^c$ ; the action of  $G^c/H_\sigma$  on the set of components of  $\mathbb{P}^\sigma$  is free.
- (5) The action of  $G^a$  on  $\mathbb{P}^\circ := \mathbb{P} \setminus \bigcup_{\sigma \in \bar{\Sigma}} \mathbb{P}^\sigma$  is free.

*Proof.* Since the order of  $G^c$  is coprime to the characteristic of  $k$ , every  $g \in G^c$  is semi-simple and we can decompose

$$V_j = \oplus_i V_j(\lambda_i(g)),$$

as a sum of eigenspaces. The subset of  $g$ -fixed points splits as a product  $\prod_{ij} \mathbb{P}(V_j(\lambda_i(g)))$ , where the product runs over different eigenvalues in different  $V_j$ . It follows that the subset of  $g$ -fixed points  $\mathbb{P}^g \subset \mathbb{P}$  is a union of products of projective subspaces of  $\mathbb{P}(V_j)$ .

If  $\sigma \in \bar{\Sigma}$  then its elements can be simultaneously diagonalized. Hence the subset of fixed points in  $\mathbb{P} = \prod_j \mathbb{P}(V_j)$  is a union of products of projective subspaces, and there is a Zariski open subvariety of  $\mathbb{P}$  stable under the action of the preimage  $\tilde{\sigma} := \pi_a^{-1}(\sigma) \subset G^c$ .

Let  $\sigma := \langle g, h \rangle \subset G^a$  be a subgroup such that  $\sigma \notin \bar{\Sigma}$ . Then the same holds for the images of  $g, h$  in  $\mathrm{GL}(V_j)$ , for at least one  $j \in J$ . Thus the commutator  $[g, h] \in \mathrm{GL}(V_j)$  is a nontrivial scalar matrix, hence they have no common eigenvectors, i.e., no common fixed points in  $\mathbb{P}(V_j)$ . Thus if  $\sigma \notin \bar{\Sigma}$  then  $\sigma$  has no fixed points in  $\prod_j \mathbb{P}(V_j)$ . Note that projective subspaces corresponding to different eigenvalues of  $g$  do not intersect in  $\mathbb{P}(V_j)$  and hence  $\mathbb{P}^\sigma$  splits into a disjoint union of products of projective subspaces of different  $\mathbb{P}(V_j)$ .

Assume that  $[h, \tilde{\sigma}] = 0$  in  $G^c$  for some  $h \in G^c$ . Then  $\langle h, \tilde{\sigma} \rangle$  has a fixed point in each component of  $\mathbb{P}^\sigma$  and  $h$  maps every component of  $\mathbb{P}^\sigma$  into itself. Thus a subgroup  $H \subset G^c$ , with  $[H, \tilde{\sigma}] = 0$  maps every component of  $\mathbb{P}^\sigma$  into itself.

Assume that  $\langle h, \gamma \rangle \notin \bar{\Sigma}$ , for some  $\gamma \in \sigma$ . Then for some  $j$ , the images of  $h, \gamma$  in  $\mathrm{GL}(V_j)$  have nonintersecting invariant subvarieties in  $\mathbb{P}(V_j)$ . In particular,  $h$  does not preserve any component of  $\mathbb{P}^\sigma$ .  $\square$

**Lemma 2.10.** *Let  $K = k(X)$  be a function field with Galois group  $G_K$ . Given a surjection  $G_K \rightarrow G^c$ , onto some finite central extension of an abelian group  $G^a$ , let  $\mathbb{P} = \prod_j \mathbb{P}(V_j)$  be the space constructed in Lemma 2.9. Then there is a rational map  $\varrho : X \rightarrow \mathbb{P}/G^a$  such that*

- $\varrho$  maps the generic point of  $X$  into  $\mathbb{P}^\circ/G^a$ ;
- the homomorphism

$$s_K : H_s^*(G^a, \mathbb{Z}/\ell^n) \rightarrow H^*(G_K, \mathbb{Z}/\ell^n)$$

*factors through the cohomology of  $\mathbb{P}^\circ/G^a$ .*

*Proof.* Let  $X^\circ \subset X$  be an open affine subvariety such that  $\pi_1(X^\circ)$  surjects onto  $G^c$ . Let  $\tilde{X}^\circ \rightarrow X^\circ$  be the induced unramified  $G^c$ -covering. Then  $k[\tilde{X}^\circ]$  decomposes into an infinite direct sum of  $G^c$ -representations.

Fix a point  $\tilde{X}^\circ$  and consider its orbit. The restriction of  $k[\tilde{X}^\circ]$  to this orbit defines a regular quotient  $G^c$ -representation isomorphic to  $k[G]$ ; this admits homomorphisms to  $k[V]$ , corresponding to maps  $X \rightarrow V/G^c$ .  $\square$

### 3. BASIC VALUATION THEORY

Let  $X$  be a variety over  $k = \bar{\mathbb{F}}_p$ ,  $K = k(X)$  its function field, and  $G_K$  the absolute Galois group of  $K$ . We write  $\mathcal{V}_K$  for the set of valuations of  $K$  and  $\mathcal{DV}_K$  for the subset of divisorial valuations. The corresponding residue fields will be denoted by  $\mathbf{K}_\nu$ . For  $\nu \in \mathcal{V}_K$ , let  $D_\nu \subset G_K$  denote a decomposition group of  $\nu$  and  $I_\nu \subset D_\nu$  the inertia subgroup; we have  $G_{\mathbf{K}_\nu} = D_\nu/I_\nu$ . The pro- $\ell$ -quotients of these groups will be denoted by  $\mathcal{G}_K$ ,  $\mathcal{D}_\nu$ , and  $\mathcal{I}_\nu$ , respectively. We will always assume that  $p \neq \ell$ . The corresponding abelianizations will be denoted by  $\mathcal{G}_K^a$ ,  $\mathcal{D}_\nu^a$ , and  $\mathcal{I}_\nu^a$ ; their canonical central extensions by  $\mathcal{G}_K^c$ ,  $\mathcal{D}_\nu^c$ , and  $\mathcal{I}_\nu^c$ . Under our assumptions,  $\mathcal{G}_K^a$  is a free  $\mathbb{Z}_\ell$ -module of infinite rank.

**Lemma 3.1.** *For  $\nu \in \mathcal{V}_K$  consider the commutative diagram*

$$\begin{array}{ccc} D_\nu & \longrightarrow & G_K \\ \pi_\nu \downarrow & & \downarrow \pi \\ \mathcal{D}_\nu^a & \xrightarrow{\delta_\nu^a} & \mathcal{G}_K^a, \end{array}$$

where  $\pi_\nu$  and  $\pi$  are the canonical projections and  $\delta_\nu^a$  is the induced homomorphism. Then  $\delta_\nu^a$  is injective with primitive image. In particular,  $\delta_\nu^a$  embeds  $\mathcal{I}_\nu^a$  as a primitive subgroup of  $\mathcal{D}_\nu^a$ .

Let  $\Sigma(\mathcal{G}_K^c)$  be the set of primitive topologically noncyclic subgroups of  $\mathcal{G}_K^c$  whose preimage in  $\mathcal{G}_K^c$  is abelian. By [9, Section 6], we have:

**Theorem 3.2.** *Assume that  $\dim(X) \geq 2$ . Then*

$$\mathrm{rk}_{\mathbb{Z}_\ell}(\sigma) \leq \dim(X), \quad \text{for all } \sigma \in \Sigma(\mathcal{G}_K^c).$$

The following key result gives a valuation-theoretic interpretation of liftable subgroups in  $\mathcal{G}_K^a$ ; it is crucial for the reconstruction of function fields in [10] and [11].

**Theorem 3.3.** [9, Corollary 6.4.4] *Assume that  $\dim(X) \geq 2$  and let  $\sigma \in \Sigma(\mathcal{G}_K^c)$ . Then there exists a valuation  $\nu \in \mathcal{V}_K$  such that  $\mathcal{I}_\nu^a$  is a subgroup of  $\sigma$  of  $\mathbb{Z}_\ell$ -corank at most one and  $\sigma \subseteq \mathcal{D}_\nu^a$ .*

We recall some background from valuation theory. For  $\nu \in \mathcal{V}_K$  let  $\Gamma_\nu = \nu(K^\times)$  be its value group. We have a fundamental inequality

$$(3.1) \quad \mathrm{tr} \deg_k(K) \geq \mathrm{tr} \deg_k(\mathbf{K}_\nu) + \dim_{\mathbb{Q}}(\Gamma_\nu \otimes \mathbb{Q}).$$

A valuation  $\nu$  is called *algebraic of rank  $r$*  if its value group  $\Gamma_\nu := \nu(K^\times)$  is isomorphic to  $\mathbb{Z}^r$  and admits a filtration

$$\mathbb{Z}^r \simeq \Gamma_\nu = \Gamma_{\nu_r} \supsetneq \dots \supsetneq \Gamma_{\nu_1} \simeq \mathbb{Z}$$

by free abelian subgroups  $\Gamma_{\nu_j}$  corank  $j$  which are value groups of embedded compatible valuations  $\nu_j$ . Such valuations arise from flags of irreducible subvarieties  $\mathfrak{c}_j$  of codimension  $j$

$$\mathfrak{c}_r \subsetneq \mathfrak{c}_{r-1} \subsetneq \dots \subsetneq \mathfrak{c}_1 \subsetneq X.$$

A valuation is called an *Abhyankar valuation* if equality holds in (3.1) (alternatively,  $\nu$  is said to be without transcendence defect); every algebraic valuation is an Abhyankar valuation.

Let  $\nu$  be an Abhyankar valuation of  $K$ . By [19, Theorem 3.4(a)],

$$\Gamma_\nu \simeq \mathbb{Z}^r,$$

for some  $r$ . By [19, Theorem 1.1], for any  $\mathcal{F} := \{f_1, \dots, f_m\} \subset \mathfrak{o}_\nu$  there exists a projective model  $X$  of  $K$  such that

- the center  $\mathfrak{c} = \mathfrak{c}_X(\nu)$  of  $\nu$  is a generically smooth subvariety of  $X$  of dimension  $\dim_{\mathbb{Q}}(\Gamma_\nu \otimes \mathbb{Q})$ ;
- there exists a regular parameter system  $(a_1, \dots, a_r)$  of  $\mathcal{O}_{X, \mathfrak{c}}$  such that each  $f_j$ ,  $j = 1, \dots, m$ , is an  $\mathcal{O}_{X, \mathfrak{c}}$ -monomial in  $\{a_1, \dots, a_r\}$ .

We say that  $\nu$  admits a smooth monomial uniformization on  $X$  with respect to  $\mathcal{F}$ . In particular, we have:

**Proposition 3.4.** *Let  $K$  be a function field of an algebraic variety over  $k = \mathbb{F}_p$ . Let  $\nu$  be an algebraic valuation of rank  $r$ . Then there exists a projective model  $X$  of  $K$  over  $k$  such that the centers  $\mathfrak{c}_j$  of  $\nu_j$  on  $X$  are irreducible subvarieties of codimension  $j$  which are smooth at the general point of  $\mathfrak{c}_r$ .*

The following statement has been proved in [20, Theorem 1.2 and Corollary 1.3].

**Proposition 3.5.** *Let  $K$  be a function field of an algebraic variety over  $k = \mathbb{F}_p$ ,  $\nu \in \mathcal{V}_K$ , and  $\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathfrak{o}_\nu$  a finite set of functions. Then there exists a finite separable Galois extension  $\tilde{K}/K$ , an extension of  $\nu$  to a valuation  $\tilde{\nu} \in \mathcal{V}_{\tilde{K}}$ , and a projective model  $\tilde{X}$  of  $\tilde{K}$  such that*

- (1) *the extension of residue fields  $\tilde{K}_{\tilde{\nu}}/\tilde{K}_{\tilde{\nu}}$  is purely inseparable;*
- (2)  *$\Gamma_{\tilde{\nu}}/\Gamma_\nu$  is a finite  $p$ -group;*
- (3)  *$\tilde{\nu}$  admits a smooth monomial uniformization on  $\tilde{X}$  with respect to  $\mathcal{F}$ .*

**Proposition 3.6.** *Let  $K$  be a function field of an algebraic variety over  $k = \bar{\mathbb{F}}_p$  and  $\mathcal{I}_\nu^a, \mathcal{D}_\nu^a \subset \mathcal{G}_K^a$  the abelianized inertia, resp. decomposition subgroup of some valuation  $\nu \in \mathcal{V}_K$ . Let*

$$\gamma_K : \mathcal{G}_K^a \rightarrow G^a$$

*be a continuous surjective homomorphism onto a finite abelian  $\ell$ -group  $G^a$ . Assume that  $\gamma_K(\mathcal{I}_\nu^a)$  is nontrivial. Then for any primitive cyclic subgroup  $I \subset \gamma_K(\mathcal{I}_\nu^a)$  there exists a divisorial valuation  $\nu' \in \mathcal{DV}_K$  such that*

$$I \subseteq \gamma_K(\mathcal{I}_{\nu'}^a) \quad \text{and} \quad \gamma_K(\mathcal{D}_\nu^a) \subseteq \gamma_K(\mathcal{D}_{\nu'}^a).$$

*Proof.* Assume that  $\ell^n$  annihilates  $G^a = \bigoplus_j \mathbb{Z}/\ell^{n_j}$ , i.e.,  $n_j \leq n$ . By Kummer theory, the homomorphism

$$\gamma_K \in \text{Hom}(\mathcal{G}_K^a/\ell^n, G^a) = \bigoplus_j \text{Hom}(\mathcal{G}_K^a, \mathbb{Z}/\ell^{n_j}) = \bigoplus_j K^\times / (K^\times)^{\ell^{n_j}}$$

is defined by a finite set of elements  $\mathcal{F} = \{f_j\}$ , with  $f_j \in K^\times / (K^\times)^{\ell^{n_j}}$ . We lift these to a set of elements of  $K^\times$ , denoted by the same letter, and we may assume that these are in  $\mathfrak{o}_\nu$ . We apply Proposition 3.5 and pass to a finite separable Galois extension  $\tilde{K}$  of  $K$  over which  $\tilde{\nu}$ , the extension of  $\nu$ , admits a smooth uniformization on a projective model  $\tilde{X}$  of  $\tilde{K}$  with respect to  $\mathcal{F}$ . Properties (1) and (2) in Proposition 3.5 insure that the image in  $G^a$  of the corresponding inertia and decomposition group is unchanged, i.e., we have a diagram

$$\begin{array}{ccccc} \mathcal{I}_{\tilde{\nu}}^a & \subseteq & \mathcal{D}_{\tilde{\nu}}^a & \subset & \mathcal{G}_{\tilde{K}}^a \\ \parallel & & \parallel & & \downarrow \\ \mathcal{I}_\nu^a & \subseteq & \mathcal{D}_\nu^a & \subset & \mathcal{G}_K^a \longrightarrow G^a \end{array}$$

Thus we may assume that  $K = \tilde{K}$ . Let  $\mathfrak{c} = \mathfrak{c}(\nu)$  be the center of the (the lift of)  $\nu$  on  $X$ . The regular parameter system  $(a_1, \dots, a_r)$  of  $\mathcal{O}_{X,\mathfrak{c}}$  defines an algebraic valuation  $\nu' = (\nu'_1, \dots, \nu'_r)$  of rank  $r$  of  $K$  and

$$\rho(\mathcal{I}_\nu^a) = \rho(\mathcal{I}_{\nu'}^a) \subseteq \rho(\mathcal{D}_\nu^a) = \rho(\mathcal{D}_{\nu'}^a).$$

We have reduced the proof to the case of algebraic valuations, where it is straightforward: If  $\rho(\mathcal{I}_{\nu'_1}^a)$  is nontrivial in  $G^a$  then we are done since

$$\rho(\mathcal{I}_{\nu'_1}^a) \subset \rho(\mathcal{I}_{\nu'}^a) = \rho(\mathcal{I}_{\nu'_1}^a) \subset \rho(\mathcal{D}_{\nu'}^a).$$

Otherwise, we can assume that  $\rho(\mathcal{I}_{\nu'_1}^a)$  and  $\rho(\mathcal{I}_{\nu'_j}^a) = 0$ ,  $j > 1$  but  $\rho(\mathcal{I}_{\nu'_i}^a) \neq 0$ . We blow up the general point of the center of  $\nu'_i$  and let  $E$  be the exceptional divisor, it is generically smooth by Proposition 3.4. Then

we have

$$0 \neq \rho(\mathcal{I}_E^a) \subset \rho(\mathcal{I}_{\nu'}^a / \mathcal{I}_{\nu'_{i+1}}^a) \subset \rho(\mathcal{D}_{\nu'}^a) / \rho(\mathcal{I}_{\nu'_{i+1}}^a) \subset \rho(\mathcal{D}_E^a)$$

and  $\rho(\mathcal{I}_E^a)$  coincides with  $\rho(\mathcal{I}_{\nu'_i}^a / \mathcal{I}_{\nu'_{i+1}}^a)$ , which is nontrivial by assumption.  $\square$

*Remark 3.7.* The proof is essentially an application of the general result that algebraic valuations (or Abhyankar valuations) are dense in the patch-topology on  $\mathcal{V}_K$ .

#### 4. LIFTABLE SUBGROUPS AND THEIR CONFIGURATIONS

Let  $K = k(X)$  be the function field of an algebraic variety over  $k = \overline{\mathbb{F}}_p$ . In this section, we compare the structure of the fan  $\Sigma(\mathcal{G}_K^c)$  with fans in its finite quotients. Consider the canonical central extension

$$(4.1) \quad 1 \rightarrow \mathcal{Z}_K \rightarrow \mathcal{G}_K^c \rightarrow \mathcal{G}_K^a \rightarrow 1.$$

**Lemma 4.1.** *We have*

$$\mathcal{Z}_K = [\mathcal{G}_K^c, \mathcal{G}_K^c].$$

*Proof.* This holds for function fields of curves since the corresponding pro- $\ell$ -quotients of their absolute Galois groups are free. In higher dimensions,  $\mathcal{G}_K^a$  embeds into the product  $\prod_E \mathcal{G}_E^a$ , where  $E$  ranges over function fields of curves  $E \subset K$ . Under the projection to  $\mathcal{G}_K^c \rightarrow \mathcal{G}_E^a$ , the center of  $\mathcal{G}_K^c$  maps to zero, hence the claim.  $\square$

**Lemma 4.2.** *Consider commutative diagrams of continuous homomorphisms*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}_K^c & \longrightarrow & \mathcal{G}_K^a \longrightarrow 1 \\ & & \downarrow & & \downarrow \gamma_K^c & & \downarrow \gamma_K \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a \longrightarrow 1, \end{array}$$

where  $G^c$  is finite, with fixed surjective  $\gamma_K$ . Assume that  $Z$  is the maximal quotient of  $\mathcal{Z}_K$  giving rise to such a diagram. Then  $G^c$  is unique, modulo isoclinism.

*Proof.* Assume that  $G_1^c, G_2^c$  are two such extensions of  $G^a$  with  $Z_1, Z_2$ , respectively, and put  $G := G_1^c \times_{G^a} G_2^c$ . We have a natural surjection  $G \rightarrow G^a$  and an inclusion  $Z_1 \times Z_2 \hookrightarrow G$ . Moreover,  $[G, G] \subseteq Z_1 \times Z_2$ . By Lemma 4.1,  $\mathcal{Z}_K$  is generated by commutators in  $\mathcal{G}_K^c$ , thus  $Z_1 \times Z_2$  is also generated by commutators in  $G$ . It follows that  $G = G^c$ . If both projections  $[G^c, G^c] \rightarrow Z_1, Z_2$  are isomorphisms then  $G_1^c$  and  $G_2^c$  are isoclinic. Otherwise, we have a contradiction to the maximality assumption. Since  $G^a$  is finite,  $G^c$  is also finite, as it is bounded by  $\wedge^2(G^a)^*$ .  $\square$



We proceed to investigate the properties of *fans* under such factorizations. Let

$$\gamma_K : \mathcal{G}_K^a \rightarrow G^a$$

be a continuous surjective homomorphism onto a finite group. We choose a maximal finite central extension  $G^c$  of  $G^a$  as in Lemma 4.2.

**Corollary 4.3.** *Given continuous surjective homomorphisms*

$$(4.2) \quad \mathcal{G}_K^a \xrightarrow{\tilde{\gamma}_K} \tilde{G}^a \xrightarrow{\gamma} G^a,$$

*with  $\tilde{G}^a$  a finite group, there is a unique (modulo isoclinism of lower rows) diagram of central extensions*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}_K^c & \longrightarrow & \mathcal{G}_K^a \longrightarrow 1 \\ & & \downarrow & & \downarrow \tilde{\gamma}_K^c & & \downarrow \tilde{\gamma}_K \\ 1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{G}^c & \longrightarrow & \tilde{G}^a \longrightarrow 1 \\ & & \downarrow & & \downarrow \gamma^c & & \downarrow \gamma \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a \longrightarrow 1 \end{array}$$

*with surjective  $\tilde{\gamma}_K^c$ ,  $\gamma^c$  and maximal  $\tilde{Z}$ ,  $Z$ .*

*Proof.* Evident. □

We will use the following observation:

**Lemma 4.4.** *Let  $\mathcal{G}^a$  be a profinite abelian group and*

$$\mathcal{G}^a \xrightarrow{\gamma_j} \tilde{G}_j^a, \quad j = 1, \dots, n,$$

*a collection of continuous surjective homomorphisms onto finite groups. Then there exists a continuous surjection*

$$\gamma : \mathcal{G}^a \rightarrow \tilde{G}^a$$

*onto a finite group such that each  $\gamma_j$  factors through  $\gamma$ :*

$$\gamma_j : \mathcal{G}^a \xrightarrow{\gamma} \tilde{G}^a \rightarrow \tilde{G}_j^a.$$

*Proof.* We can choose  $\tilde{G}^a$  to be the image of  $\mathcal{G}^a$  in the direct product

$$\tilde{G}_1 \times \dots \times \tilde{G}_n.$$

□

We are interested in factorizations (4.2), with finite  $\tilde{G}^a$ , preserving liftable subgroups and their configurations. Throughout we will be

working with the canonical, modulo isoclinism, diagram as in Corollary 4.3, i.e., a factorization as in Equation (4.2) will canonically determine  $\Sigma(\tilde{G}^c)$  and the set of  $\Delta$ -pairs in  $\tilde{G}^a$ , by Lemma 2.1. Let

$$\Sigma_E(G^c) := \{\sigma \in \Sigma(G^c) \mid \sigma = \gamma_K(\sigma_K), \quad \text{for some } \sigma_K \in \Sigma(\mathcal{G}_K^c)\}$$

be the subset of *extendable* subgroups.

**Lemma 4.5.** *Given a continuous surjective homomorphism*

$$\gamma_K : \mathcal{G}_K^a \rightarrow G^a$$

*onto a finite abelian group there exists a factorization*

$$\mathcal{G}_K^a \xrightarrow{\tilde{\gamma}_K} \tilde{G}^a \xrightarrow{\gamma} G^a, \quad \gamma_K = \gamma \circ \tilde{\gamma}_K,$$

*with finite  $\tilde{G}^a$ , such that for all  $\sigma \in \Sigma(G^c)$  we have: if  $\sigma$  is nonextendable then there is no  $\tilde{\sigma} \in \Sigma(\tilde{G}^c)$  with  $\gamma(\tilde{\sigma}) = \sigma$ .*

*Moreover, this holds for every finite  $\bar{G}^a$  fitting into*

$$\mathcal{G}_K^a \rightarrow \bar{G}^a \rightarrow \tilde{G}^a.$$

*Proof.* First we prove the statement for one nonextendable  $\sigma$ . Write

$$(4.3) \quad \mathcal{G}_K^a = \text{proj} \lim_{\iota \in I} G_\iota^a, \quad \gamma_{\iota\iota'} : G_\iota^a \longrightarrow G_{\iota'}^a, \quad \iota' \preceq \iota,$$

where the limit is over finite continuous quotients of  $\mathcal{G}_K^a$ . Assume that for all  $\iota$ , there is some  $\sigma_\iota \in \Sigma(G_\iota^c)$  surjecting onto  $\sigma$ ; this implies that there exist such  $\sigma_{\iota'}$ , for all  $\iota' \preceq \iota$ , with  $\gamma_{\iota\iota'}(\sigma_\iota) = \sigma_{\iota'}$ .

By compactness of  $\mathcal{G}_K^a$ , there exists a closed liftable  $\sigma_K \subset \mathcal{G}_K^a$  surjecting onto  $\sigma$ . This contradicts our assumption that  $\sigma$  is nonextendable. Thus there is a required factorization

$$\mathcal{G}_K^a \rightarrow \tilde{G}^a \xrightarrow{\gamma} G^a.$$

Let

$$\{\sigma_1, \dots, \sigma_n\} = \Sigma(G^c) \setminus \Sigma_E(G^c).$$

For each  $j$ , let

$$\mathcal{G}_K^a \longrightarrow \tilde{G}_j^a \xrightarrow{\gamma_j} G^a$$

be the factorization constructed above. Now we apply Lemma 4.4, combined with Corollary 4.3, and obtain factorizations of  $\gamma_j$ :

$$\mathcal{G}_K^a \rightarrow \tilde{G}^a \rightarrow \tilde{G}_j \rightarrow G^a, \quad \gamma : \tilde{G}^a \rightarrow G^a,$$

Assume that there is some  $j$  for which there exists a  $\tilde{\sigma} \in \Sigma(\tilde{G}^c)$  surjecting onto  $\sigma_j$ . Then image  $\tilde{\sigma}$  in  $\tilde{G}_j$  must be liftable, contradicting the construction in the first part.  $\square$

Let

$$I \subseteq D \subseteq G^a.$$

be a  $\Delta$ -pair (see Definition 1.4). Throughout, we assume that  $G^a$  arises as a finite quotient of the Galois group of some function field  $G_K$ , in particular, the corresponding  $G^c$  is determined as in Lemma 4.2, up to isoclinism. We say that  $(I, D)$  is *extendable* if there exists a valuation  $\nu \in \mathcal{V}_K$  and subgroups

$$I^a \subseteq \mathcal{I}_\nu^a, \quad D^a \subseteq \mathcal{D}_\nu^a$$

such that

$$\gamma_K(I^a) = I, \quad \gamma_K(D^a) = D.$$

Recall that a  $\Delta$ -pair  $(\tilde{I}, \tilde{D})$  is said to surject onto  $(I, D)$  if

$$\gamma(\tilde{I}) = I, \quad \gamma(\tilde{D}) = D.$$

We will need the following strengthening of Lemma 4.5

**Proposition 4.6.** *Given a continuous surjective homomorphism*

$$\mathcal{G}_K^a \rightarrow G^a$$

*onto a finite abelian group there exists a factorization*

$$\mathcal{G}_K^a \rightarrow \tilde{G}^a \rightarrow G^a,$$

*with finite  $\tilde{G}^a$ , such that for all  $\Delta$ -pairs  $(I, D)$  in  $G^a$  we have: if  $(I, D)$  nonextendable then there is no  $\Delta$ -pair  $(\tilde{I}, \tilde{D})$  in  $\tilde{G}^a$  surjecting onto  $(I, D)$ .*

*Proof.* As in the proof of Lemma 4.5, it suffices to establish the statement for one nonextendable  $\Delta$ -pair; indeed, there are only finitely many  $\Delta$ -pairs in  $G^a$  and the same application of Lemma 4.4 will then establish it for all.

Assume that there is no finite quotient of  $\mathcal{G}_K^a$  with the desired property. We start with a factorization

$$\mathcal{G}_K^a \rightarrow \tilde{G}^a \xrightarrow{\gamma} G^a$$

such that  $\tilde{G}^a$  satisfies the conclusions of Lemma 4.5, i.e., no  $\sigma \in \Sigma(G^c) \setminus \Sigma_E(G^c)$  is the image of a  $\tilde{\sigma} \in \Sigma(\tilde{G}^c)$ .

Let  $(I, D)$  be a nonextendable  $\Delta$ -pair. By our assumption, there exists a  $\Delta$ -pair  $(\tilde{I}, \tilde{D})$  in  $\tilde{G}^a$  surjecting onto  $(I, D)$ . Choose representatives  $g_1, \dots, g_n \in D$  for  $D/I$  and their preimages  $\tilde{g}_j := \gamma^{-1}(g_j) \in \tilde{D}$ . Note that for each  $j$ ,

$$\sigma_j := \langle g_j, I \rangle \in \Sigma(G^c), \quad \tilde{\sigma}_j := \langle \tilde{g}_j, \tilde{I} \rangle \in \Sigma(\tilde{G}^c).$$

and that  $\tilde{\sigma}_j$  surject onto  $\sigma_j$ . By Lemma 4.5 and our choice of  $\tilde{G}^a$ , all  $\sigma_j$  are extendable. Moreover,

$$\tilde{I} = \cap_{j=1}^n \tilde{\sigma}_j.$$

We replace and rename the original  $\tilde{D}$  by

$$\tilde{D} := \cup_{j=1}^n \tilde{\sigma}_j.$$

Then  $(\tilde{I}, \tilde{D})$  is a  $\Delta$ -pair in  $\tilde{G}^a$  surjecting onto  $(I, D)$ .

Now we consider a projective system of finite continuous quotients

$$\mathcal{G}_K^a \rightarrow \tilde{G}_\iota^a \rightarrow \tilde{G}^a \rightarrow G^a, \quad \gamma_{\iota\iota'} : G_\iota^a \rightarrow G_{\iota'}^a, \quad \iota' \preceq \iota.$$

Assume that for each  $\iota$  there exists a  $\Delta$ -pair  $(\tilde{I}_\iota, \tilde{D}_\iota)$  in  $\tilde{G}_\iota^a$  surjecting onto  $(I, D)$ . Iterating the construction above, we construct, for each  $\iota$ , a collection of liftable subgroups

$$\tilde{\sigma}_{\iota,1}, \dots, \tilde{\sigma}_{\iota,n}$$

and a  $\Delta$ -pair  $(\tilde{I}_\iota, \tilde{D}_\iota)$  of the form

$$\tilde{I}_\iota = \cap_{j=1}^n \tilde{\sigma}_{\iota,j}, \quad \tilde{D}_\iota := \cup_{j=1}^n \tilde{\sigma}_{\iota,j},$$

such that

- $(\tilde{I}_\iota, \tilde{D}_\iota)$  surjects onto  $(\tilde{I}_{\iota'}, \tilde{D}_{\iota'})$ , for each  $\iota' \preceq \iota$ ,

and in particular onto  $(I, D)$ . By compactness of  $\mathcal{G}_K^a$  (see Lemma 4.5), there exist closed subgroups

$$\sigma_{K,1}, \dots, \sigma_{K,n} \in \Sigma(\mathcal{G}_K^c);$$

the closed subgroups

$$\mathcal{I}^a := \cap \sigma_{K,j}, \quad \mathcal{D}^a := \cup \sigma_{K,j}$$

of  $\mathcal{G}_K^a$  surject onto  $I$ , resp.  $D$ . On the other hand, by the theory of liftable pairs (see [9, Section 6] and [11, Corollary 4.3]), there exists a valuation  $\nu \in \mathcal{V}_K$  such that

$$\mathcal{I}^a \subseteq \mathcal{I}_\nu^a, \quad \mathcal{D}^a \subseteq \mathcal{D}_\nu^a.$$

This contradicts our assumption that  $(I, D)$  is not extendable.  $\square$

## 5. GALOIS COHOMOLOGY OF FUNCTION FIELDS

In [10], [11] we proved that if  $k = \bar{\mathbb{F}}_p$ , with  $p \neq \ell$ , and  $X$  is an algebraic variety over  $k$  of dimension  $\geq 2$  then  $K = k(X)$  is encoded, up to purely inseparable extensions, by  $\mathcal{G}_K^c$ , the second lower series quotient of  $\mathcal{G}_K$ . Related reconstruction results have been obtained in [24], [23], [25].

The proof of the Bloch–Kato conjecture by Voevodsky, Rost, and Weibel, substantially advanced our understanding of the relations between fields and their Galois groups, in particular, their Galois cohomology. Indeed, consider the diagram

$$\begin{array}{ccc} & G_K & \\ \pi_c \swarrow & & \searrow \pi \\ \mathcal{G}_K^c & \xrightarrow{\pi_a} & \mathcal{G}_K^a. \end{array}$$

The following theorem relates the Bloch–Kato conjecture to statements in Galois-cohomology, with coefficients in  $\mathbb{Z}/\ell^n$  (see also [13], [14], [26]).

**Theorem 5.1.** [3], [12, Theorem 11] *Let  $k = \bar{\mathbb{F}}_p$ ,  $p \neq \ell$ , and  $K = k(X)$  be the function field of an algebraic variety of dimension  $\geq 2$ . The Bloch–Kato conjecture for  $K$  is equivalent to:*

(1) *The map*

$$\pi^* : H^*(\mathcal{G}_K^a, \mathbb{Z}/\ell^n) \rightarrow H^*(G_K, \mathbb{Z}/\ell^n)$$

*is surjective and*

(2)  $\text{Ker}(\pi_a^*) = \text{Ker}(\pi^*)$ .

This implies that the Galois cohomology of the pro- $\ell$ -quotient  $\mathcal{G}_K$  of the absolute Galois group  $G_K$  encodes important birational information of  $X$ . For example, in the case above,  $\mathcal{G}_K^c$ , and hence  $K$ , modulo purely-inseparable extensions, can be recovered from the cup-products

$$H^1(\mathcal{G}_K, \mathbb{Z}/\ell^n) \cup H^1(\mathcal{G}_K, \mathbb{Z}/\ell^n) \rightarrow H^2(\mathcal{G}_K, \mathbb{Z}/\ell^n), \quad n \in \mathbb{N}.$$

From now on, we will frequently omit the coefficient ring  $\mathbb{Z}/\ell^n$  from notation.

The first part of the Bloch–Kato theorem says that every  $\alpha_K \in H^i(G_K)$  is induced from a cohomology class  $\alpha^a \in H^i(G^a)$  of some finite abelian quotient  $G_K \rightarrow G^a$ . An immediate application of this is the following proposition:

**Proposition 5.2.** *Let  $\alpha_K \in H^i(G_K)$  be defined on a model  $X$  of  $K$  and induced from a continuous surjective homomorphism  $\chi : \hat{\pi}_1(X) \rightarrow G$  onto a finite group. Let  $\alpha = \alpha_X \in H_{\text{et}}^i(X)$  be the class representing  $\alpha_K$  on  $X$ . Then there exists a finite cover  $X = \cup_j X_j$  by Zariski open subvarieties such that, for each  $j$ , the restriction  $\alpha_j := \alpha|_{X_j}$  is induced from a continuous surjective homomorphism  $\chi_j : \hat{\pi}_1(X_j) \rightarrow G^a$  onto a finite abelian group and a class  $\alpha^a \in H_s^i(G^a) = \wedge^i(H^1(G^a))$ .*

*Proof.* We first apply the Bloch-Kato theorem to  $V^\circ/G$  and find a Zariski open subset  $U := U_\alpha^\circ$  such that the restriction  $\alpha_U$  is as claimed, i.e., induced from a class  $\alpha^a \in H_s^i(G^a) = \wedge^i(H^1(G^a))$ , for some homomorphism  $\chi_a : G_K \rightarrow G^a$  to a finite abelian group.

By Lemma 1.1, for every  $x \in X$  there exists a map  $f = f_x$  such that  $f(x) \subset U$  and the restriction of  $\alpha$  to  $f^{-1}(U) \subset X$  equals  $f^*(\chi_a^*(\alpha^a))$ . The claim follows by choosing a finite cover by open subvarieties with these properties.  $\square$

The second part implies the following:

**Corollary 5.3.** *Let  $\alpha_K \in H^i(G_K)$ . Assume that we are given finitely many quotients*

$$\chi_j : G_K \rightarrow G_j^a$$

*onto finite abelian groups and classes*

$$\alpha_j^a \in H^i(G_j^a)$$

*with  $\chi_j^*(\alpha_j^a) = \alpha_K$ , for all  $j$ . Then there exists a continuous finite quotient  $G_K \rightarrow G^c$  onto a finite central extension of an abelian group  $G^a$  such that*

- $\chi_j$  factor through  $G^c$ , i.e., there exist surjective homomorphisms  $\psi_j : G^c \rightarrow G_j^a$ , for all  $j$ ;
- there exists a class  $\alpha^c \in H^i(G^c)$  with

$$\alpha^c = \psi_j^*(\alpha_j^a), \quad \text{for all } j.$$

**Lemma 5.4.** *Let  $Z \subset X$  be an affine subset,  $Z = \text{Spec}(\mathfrak{o}_Z)$ , and  $\nu \in \mathcal{V}_K$ . The general point of  $\nu$  is contained in  $Z$ ,*

$$\mathfrak{c}_X(\nu)^\circ \subseteq Z$$

*if and only if  $\nu(f) \geq 0$ , for all  $f \in \mathfrak{o}_Z$ .*

**Lemma 5.5.** *Let  $X$  be a normal variety with function field  $K$ . Assume that  $\alpha_K \in H^i(G_K)$  is defined on  $X$  and induced from a homomorphism  $\chi : \hat{\pi}_1(X) \rightarrow G$  to a finite group  $G$ . Consider the sequence*

$$\chi_K : G_K \rightarrow \hat{\pi}_1(X) \xrightarrow{\chi} G.$$

*Then  $\chi_K(I_\nu) = 0$ , for every  $\nu$  such that  $\mathfrak{c}_X(\nu)^\circ \subset X$ .*

*Proof.* An étale cover of  $X$  induces an étale cover of the generic point of  $\mathfrak{c}_X(\nu)$ , thus the cover is unramified in  $\nu$ , i.e.,  $\chi_K(I_\nu) = 0$ .  $\square$

**Corollary 5.6.** *Let  $\alpha_K \in H^i(G_K)$ . Let  $X$  be a normal projective model of  $K$  and  $\cup_j X_j$  a finite cover by open subvarieties such that  $\alpha_K$  is defined on  $X_j$ , for each  $j$ , and is induced from a class  $\alpha_j^a \in H^i(G_j^a)$ , via a homomorphism  $\chi_j : \hat{\pi}_1(X_j) \rightarrow G_j^a$  to some finite abelian group.*

Then there exists a quotient  $\chi_K : G_K \rightarrow G^a$  and a class  $\alpha^a \in H^i(G^a)$  such that

- $\alpha^a$  induces  $\alpha_K$  and
- $\alpha^a$  is unramified on every extendable  $\Delta$ -pair  $(I, D)$  in  $G^a$ .

*Proof.* Each  $\alpha_j^a$  is unramified on all  $\nu$  such that the generic point  $\mathfrak{c}_X(\nu)^\circ \subset X_j$ , by Lemma 5.5. Since the classes  $\alpha_j$  define the same element  $\alpha_K \in H^i(G_K)$ , there exists an intermediate quotient  $G^a$  and a class  $\alpha^a \in H^i(G^a)$  inducing  $\alpha_K$ .  $\square$

## 6. UNRAMIFIED COHOMOLOGY

An important class of birational invariants of algebraic varieties are *unramified* cohomology groups, with finite constant coefficients (see [4], [16]). These are defined as follows: Let  $\nu$  be a divisorial valuation of  $K$ . We have a natural homomorphism

$$\partial_\nu : H^i(G_K) \rightarrow H^{i-1}(G_{\mathbf{K}_\nu}).$$

Classes in  $\ker(\partial_\nu)$  are called *unramified with respect to  $\nu$* . The *unramified* cohomology is

$$H_{nr}^i(G_K) := \bigcap_{\nu \in \mathcal{DV}_K} \ker(\partial_\nu) \subset H^i(G_K).$$

For  $i = 2$  this is the *unramified Brauer group* which was used to provide counterexamples to Noether's problem, i.e., nonrational varieties of type  $V/G$ , where  $V$  is a faithful representation of a finite group  $G$  (see [27], [2]).

Generally, for  $\nu \in \mathcal{V}_K$  and  $\alpha \in H^i(G_K)$  let

$$\alpha_\nu \in H^i(D_\nu)$$

be the restriction of  $\alpha$  to the decomposition subgroup  $D_\nu \subset G_K$  of  $\nu$ .

**Lemma 6.1.** *A class  $\alpha \in \ker(\partial_\nu) \subseteq H^i(G_K)$ , for  $\nu \in \mathcal{DV}_K$ , if and only if  $\alpha_\nu$  is induced from the quotient  $G_{\mathbf{K}_\nu} = D_\nu/I_\nu$ . In particular,  $\alpha_\nu$  is well-defined as an element in  $H^i(G_{\mathbf{K}_\nu})$ .*

*Proof.* The exact sequence

$$1 \rightarrow I_\nu \rightarrow D_\nu \rightarrow G_{\mathbf{K}_\nu} \rightarrow 1$$

admits a noncanonical splitting, i.e.,  $D_\nu$  is noncanonical direct product of  $G_{\mathbf{K}_\nu} = D_\nu/I_\nu$  with the corresponding inertia group, which is a torsion-free central procyclic subgroup of  $D_\nu$ . Thus

$$H^*(D_\nu) = H^*(G_{\mathbf{K}_\nu}) \otimes \wedge^* H^1(I_\nu).$$

We have

$$H^1(I_\nu, \mathbb{Z}/\ell^n) = H^0(I_\nu, \mathbb{Z}/\ell^n) = \mathbb{Z}/\ell^n$$

and

$$\wedge^*(H^1(I_\nu, \mathbb{Z}/\ell^n)) = H^1(I_\nu, \mathbb{Z}/\ell^n) \oplus H^0(I_\nu, \mathbb{Z}/\ell^n).$$

Thus

$$H^i(D_\nu) = H^{i-1}(G_{\mathbf{K}_\nu}^a) \otimes \wedge^*(H^1(I_\nu)) \oplus H^i(G_{\mathbf{K}_\nu}^a)$$

and the differential  $\partial_\nu$  coincides with the projection onto the first summand. Hence  $\partial_\nu(\alpha) = 0$  is equivalent to  $\alpha_\nu$  being induced from  $G_{\mathbf{K}_\nu}^a = D_\nu/I_\nu$ .  $\square$

Let  $K$  be a function field over  $k = \bar{\mathbb{F}}_p$ . By Theorem 5.1, a class  $\alpha \in H^i(G_K)$  is induced from  $\alpha_K^a \in \mathcal{G}_K^a$ , which in turn is induced from a finite quotient  $\mathcal{G}_K^a \rightarrow G^a$ . Applying Proposition 3.6 we obtain an alternative characterization unramified classes.

**Corollary 6.2.** *A class  $\alpha_K \in H^i(G_K)$  is unramified if and only if for every  $\nu \in \mathcal{V}_K$  the restriction of  $\alpha_K^c := \pi_a^*(\alpha_K^a) \in H^i(\mathcal{G}_K^c)$  to  $\mathcal{D}_\nu^c$  is induced from  $\mathcal{D}_\nu^c/\mathcal{I}_\nu^c$ .*

*Proof.* Assuming that for every  $\nu \in \mathcal{DV}_K$ , the restriction of  $\alpha_K^a$  to  $\mathcal{D}_\nu^a$  is induced from  $\mathcal{D}_\nu^a/\mathcal{I}_\nu^a$  we need to show this property for every  $\nu$ . This follows from Proposition 3.6.  $\square$

This allows to extend the notion of *unramified* to arbitrary valuations.

Combining the considerations above we obtain the notion of *unramified stable cohomology*

$$H_{s,nr}^*(G)$$

of a finite group  $G$ : a stable cohomology class  $\alpha \in H_s^i(G)$  is unramified if and only if it is contained in the kernel of the composition

$$H_s^i(G) \rightarrow H^i(G_K) \xrightarrow{\partial_\nu} H^{i-1}(G_{\mathbf{K}_\nu}),$$

for every valuation  $\nu \in \mathcal{DV}_K$ , where  $K = k(V/G)$  for some faithful representation of  $G$ . This does not depend on the choice of  $V$ , provided  $\ell \neq \text{char}(k)$ . These groups are contravariant in  $G$  and form a subring

$$H_{s,nr}^*(G) \subset H_s^*(G).$$

Furthermore:

- If  $V/G$  is stably rational then  $H_{s,nr}^i(G) = 0$ , for all  $i \geq 2$ .
- We have

$$H_{s,nr}^i(G) \subseteq H_{s,nr}^i(\text{Syl}_\ell(G))^{N_G(\text{Syl}_\ell(G))}.$$



*Remark 6.3.* In [8], we proved that  $H_{s,nr}^i(G) = 0$ ,  $i \geq 1$ , for most quasi-simple groups of Lie type. A complete result for quasi-simple groups and  $i = 2$  was obtained in [21].

Note that the  $\ell$ -Sylow-subgroups of finite simple groups often have stably-rational fields of invariants; this provides an alternative approach to our vanishing theorem.

**Lemma 6.4.** *Let*

$$1 \rightarrow Z \rightarrow G^c \xrightarrow{\pi} G^a \rightarrow 1$$

*be a central extension of finite  $\ell$ -groups and  $\alpha^a \in H_s^2(G^a)$ . Then  $\alpha^c := \pi_a^*(\alpha) \in H_{s,nr}^2(G^c)$  if and only if for all  $\sigma \in \Sigma := \Sigma(G^c)$  the restriction of  $\alpha^a$  to  $\sigma$  is trivial, i.e.,  $\alpha^a \in R^2(\Sigma)$ .*

**Lemma 6.5.** *Let*

$$\gamma_K : \mathcal{G}_K^a \rightarrow G^a$$

*be a continuous surjective homomorphism and  $\alpha_K^a = \gamma_K^*(\alpha^a) \in H^i(\mathcal{G}_K^a)$ , for some  $\alpha^a \in H_s^i(G^a)$ . Assume that  $\alpha_K^a$  is unramified and let  $(I, D)$  be an extendable  $\Delta$ -pair in  $G^a$ . Then  $\alpha^a$  is unramified with respect to  $(I, D)$ .*

*Conversely, if  $\alpha^a$  is unramified with respect to every extendable  $\Delta$ -pair in  $G^a$  then  $\alpha_K^a \in H_{nr}^i(\mathcal{G}_K^a)$ .*

*Proof.* For the converse, for  $\nu \in \mathcal{DV}_K$ , let  $D := \gamma_K(\mathcal{D}_\nu^a)$  and  $I := \gamma_K(\mathcal{I}_\nu^a)$ . Then either  $D$  is cyclic or  $(D, I)$  is an extendable  $\Delta$ -pair in  $G^a$ . We have a commutative diagram

$$\begin{array}{ccccc} H_s^i(G^a) & \xrightarrow{\phi} & H_s^i(D) & \xleftarrow{\psi} & H_s^i(D/I) \\ \gamma_K^* \downarrow & & \downarrow & & \downarrow \\ H^i(\mathcal{G}_K^a) & \longrightarrow & H^i(\mathcal{D}_\nu^a) & \longleftarrow & H^i(\mathcal{D}_\nu^a/\mathcal{I}_\nu^a) \end{array}$$

In either case,  $\alpha_K^a$  is unramified with respect to  $\nu$ , by Lemma 1.7.  $\square$

## 7. MAIN THEOREM

**Theorem 7.1.** *Let  $K = k(X)$  be a function field over  $k = \bar{\mathbb{F}}_p$  of  $\text{tr deg}_k(K) \geq 2$  and  $\alpha_K \in H_{nr}^i(G_K)$ , with  $\ell \neq p$  and  $i > 1$ . Then there exist a continuous homomorphism  $G_K \rightarrow \tilde{G}^a$  onto a finite  $\ell$ -group, fitting into a diagram*

$$\begin{array}{c} G_K \\ \downarrow \\ 1 \longrightarrow \tilde{Z} \longrightarrow \tilde{G}^c \longrightarrow \tilde{G}^a \longrightarrow 1 \end{array}$$

*and a class  $\tilde{\alpha}^a \in H^i(\tilde{G}^a)$  such that*

- (1)  $\alpha_K$  is induced from  $\tilde{\alpha}^a$ ,
- (2)  $\tilde{\alpha}^c := \pi_a^*(\tilde{\alpha}^a) \in H_{s,nr}^*(\tilde{G}^c)$ .

Conversely, every  $\alpha_K \in H^i(G_K)$  induced from  $\wedge^*(H^1(\tilde{G}^a))$  and unramified on some  $\tilde{G}^c$  as above is in  $H_{nr}^i(G_K)$ .

In this section we begin the proof of Theorem 7.1, reducing it to geometric statements addressed in Sections 8 and 9.

Fix an unramified class

$$\alpha_K \in H_{nr}^i(G_K) \subset H^i(G_K).$$

By Theorem 5.1, we have a surjection

$$\pi^* : H^i(\mathcal{G}_K^a) \rightarrow H^i(G_K),$$

let  $\alpha_K^a \in H^i(\mathcal{G}_K^a)$  be the unramified class such that  $\pi^*(\alpha_K^a) = \alpha_K$ . Let

$$\gamma_K : \mathcal{G}_K^a \rightarrow G^a$$

be a continuous quotient onto a finite abelian  $\ell$ -group such that  $\alpha_K^a$  is induced from a class  $\alpha^a \in H_s^i(G^a) = \wedge^i(H^1(G^a))$ . We have a diagram of central extensions:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}_K^c & \longrightarrow & \mathcal{G}_K^a \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \gamma_K \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \xrightarrow{\pi_a} & G^a \longrightarrow 1 \end{array}$$

where the lower row is uniquely defined, up to isoclinism, as in Lemma 4.2. The group  $G^a$  might be too small, i.e., it may happen that

$$\pi_a^*(\alpha^a) \notin H_{s,nr}^i(G^c).$$

Our goal is to pass to an intermediate finite quotient  $\mathcal{G}_K^a \rightarrow \tilde{G}^a$  fitting into a commutative diagram below

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{Z}_K & \longrightarrow & \mathcal{G}_K^c & \longrightarrow & \mathcal{G}_K^a \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \tilde{\gamma}_K \\ 1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{G}^c & \longrightarrow & \tilde{G}^a \longrightarrow 1 \\ & & & & \downarrow & & \downarrow \gamma \\ 1 & \longrightarrow & Z & \longrightarrow & G^c & \longrightarrow & G^a \longrightarrow 1 \end{array}$$

where the vertical arrows are surjections onto finite  $\ell$ -groups, and such that  $\alpha_K^a$  is induced from a class  $\tilde{\alpha}^a \in H^*(\tilde{G}^a)$  with

$$\tilde{\pi}_a^*(\tilde{\alpha}^a) \in H_{s,nr}^i(\tilde{G}^c).$$

There are two possibilities:

- (1) There exists a finite quotient  $G_K \rightarrow G^a$  such that  $\alpha_K$  is induced from  $\alpha^a \in H_s^i(G^a)$  which is unramified with respect to every extendable  $\Delta$ -pair  $(I, D)$  in  $G^a$ . This case is treated in Lemma 7.2.
- (2) On *every* finite quotient, the class  $\alpha^a$  inducing  $\alpha_K$  is ramified on *some* extendable  $\Delta$ -pair  $(I, D)$ . This possibility is eliminated by Lemma 7.3.

**Lemma 7.2.** *Assume that  $\alpha^a \in H_s^i(G^a)$  is unramified with respect to every extendable  $\Delta$ -pair  $(I, D)$  in  $G^a$ . Then there is a factorization*

$$(7.1) \quad \mathcal{G}_K^a \xrightarrow{\tilde{\gamma}_K} \tilde{G}^a \xrightarrow{\gamma} G^a, \quad \gamma_K = \gamma \circ \tilde{\gamma}_K,$$

with finite  $\tilde{G}^a$ , such that

$$\pi_a^*(\tilde{\alpha}^a) \in H_{s,nr}^i(\tilde{G}^c), \quad \tilde{\alpha}^a := \gamma^*(\alpha^a).$$

*Proof.* Let  $\tilde{G}^a$  be the quotient constructed in Proposition 4.6, i.e., if  $(I, D)$  is not an extendable  $\Delta$ -pair in  $G^a$  then *no*  $\Delta$ -pair  $(\tilde{I}, \tilde{D})$  surjects onto  $(I, D)$ . Thus, for each  $\Delta$ -pair  $(\tilde{I}, \tilde{D})$  in  $\tilde{G}^a$  one of the following holds:

- either  $\gamma(\tilde{I}) = 0$  or  $\gamma(\tilde{D})$  cyclic,
- or  $(\gamma(\tilde{I}), \gamma(\tilde{D}))$  is an extendable  $\Delta$ -pair in  $G^a$ .

By Lemma 1.7, combined with Lemma 6.5,  $\tilde{\alpha}^a := \gamma^*(\alpha^a)$  is unramified with respect to  $(I, D)$ .  $\square$

**Lemma 7.3.** *There exists a finite quotient  $G_K \rightarrow G^a$  such that  $\alpha^a \in H^i(G^a)$  induces  $\alpha_K$  and is unramified on every extendable  $\Delta$ -pair  $(I, D)$  in  $G^a$ .*

The proof of this lemma is presented in Section 8, in the case when  $K$  admits a smooth projective model; a reduction to the smooth case is postponed until Section 9.

## 8. THE SMOOTH CASE

Let  $X$  be a smooth projective irreducible variety over an algebraically closed field  $k$  with function field  $K = k(X)$ . By the Bloch-Ogus theorem, there is an isomorphism

$$H_{nr}^i(G_K) = H_{Zar}^0(X, \mathcal{H}_{et}^i(X)),$$

where  $\mathcal{H}_{et}^i$  is an étale cohomology sheaf (see also Theorem 4.1.1 in [15]). In particular, a class  $\alpha_K \in H_{nr}^i(G_K)$  can be represented by a finite collection of classes  $\{\alpha_n\}_{n \in N}$ , with  $\alpha_n$  defined on some Zariski open affine  $X_n \subset X$ , with  $X = \bigcup_n X_n$ , such that the restrictions of  $\alpha_n$  to some common open affine subvariety  $X^\circ \subset \bigcap_n X_n$  coincide. We will need the following strengthening:

**Lemma 8.1.** *Let  $X$  be a smooth variety with function field  $K$  and  $S = \{x_1, \dots, x_r\} \subset X$  a finite set of points. Given a class  $\alpha_K \in H_{nr}^i(G_K)$  there exist*

- a Zariski open subset  $U_S \subset X$ , containing  $S$ , and
- a class  $\alpha_S \in H_{et}^i(U_S)$

such that  $\alpha_K$  and  $\alpha_S$  coincide on some dense Zariski open subset  $U_S^\circ \subset U_S$ , i.e., for every representation of  $\alpha_K$  by  $\{\alpha_n\}_{n \in N}$  as above there exists some dense Zariski open  $U_S^\circ \subset U_S$  such that the restrictions of  $\alpha_S$  and all  $\alpha_n$  to  $U_S^\circ$  coincide.

*Proof.* Fix a representation of  $\alpha_K$  by  $\{\alpha_n\}_{n \in N}$  as above; in particular  $\alpha_n - \alpha_{n'} = 0$  on  $X_n \cap X_{n'}$ , for all  $n, n' \in N$ . Every point in  $S$  is contained in some  $X_n$ ; refining the cover by Zariski open subsets, we can achieve that each point in  $S$  is contained in exactly one such subset; i.e., after relabeling, we may assume that  $x_n \subset X_n$ , for  $n = 1, \dots, r$ . Put

$$X_S^\circ := \bigcap_{n=1}^r X_n \quad \text{and} \quad X_n^\circ := X_n \setminus [((X \setminus X_n) \cup (X \setminus X_{n'})) \cap (X \setminus X_S^\circ)].$$

If  $x_n \in X_n$  then  $x_n \in X_n^\circ$ . Moreover,

$$X_S^\circ = X_n^\circ \cap X_{n'}^\circ, \quad \text{for all } n, n' \in [1, \dots, r].$$

Define

$$U_S := \bigcup_{n=1}^r X_n^\circ.$$

The restrictions of  $\alpha_n$  to  $X_n^\circ$  can be glued to a unique class  $\alpha_S \in H_{et}^i(U_S)$ , via the Mayer-Vietoris exact sequence. This is the required class.  $\square$

Fix a representation of  $\alpha_K \in H_{nr}^i(G_K)$  by  $\{\alpha_n\}_{n \in N}$  as above. Each class  $\alpha_n \in H_{et}^i(X_n)$  is represented by a finite collection  $\{X_{nm}\}$  of affine charts  $X_{nm}$ , with  $\bigcup X_{nm} = X_n$  and finite étale covers

$$\psi_{nm} : \tilde{X}_{nm} \rightarrow X_{nm},$$

such that the restrictions  $\alpha_{nm} := \alpha_n|_{X_{nm}}$  are induced from homomorphisms  $\chi_{nm} : \hat{\pi}_1(X_{nm}) \rightarrow G_{nm}$  onto finite groups. Proposition 5.2 implies that there is further refinement of the cover by affine subcovers

$$X = \cup_j X_j,$$

such that for each  $j$  there exist

- a finite *abelian* group  $G_j^a$
- a surjection  $\chi_{K,j} : G_K \rightarrow G_j^a$ , and
- a class  $\alpha_j \in H^i(G_j^a)$  inducing  $\alpha$  via  $\chi_{K,j}$ .

Corollary 5.3 implies that there exists a finite quotient  $\pi_c : G_K \rightarrow G^c$  onto a central extension of an abelian group  $G^a$  such that the projections  $\chi_{K,j}$  factor through  $G^c$  and the images of  $\alpha_j$  in  $H^i(G^c)$  coincide. In particular,  $\alpha_K$  is induced from  $\alpha^c \in H^i(G^c)$ .

We claim that  $\alpha^c$  is unramified on every pair  $(\pi_c(I_\nu), \pi_c(D_\nu))$ , for  $\nu \in \mathcal{V}_K$ . Indeed, for each  $\nu$ ,  $\chi_{K,j}(I_\nu) = 0$  on at least *one* of the charts  $X_j$ , thus  $\alpha_j$  is induced from  $\chi_{K,j}(D_\nu)/\chi_{K,j}(I_\nu)$ ; since  $\alpha^c \in H^i(G^c)$  is induced from  $\alpha_j$ , we have the same property for  $\alpha^c$ , with respect to the pair  $(\pi_c(I_\nu), \pi_c(D_\nu))$ .

The rest of the argument is similar to the proof of Lemma 7.2. Let  $G_K \rightarrow G^a$  be an intermediate quotient surjecting onto each  $G_j^a$  and

$$G_K \rightarrow \tilde{G}^a \rightarrow G^a$$

the intermediate finite quotient constructed in Proposition 4.6. In particular, the projection of every  $\Delta$ -pair in  $\tilde{G}^a$  to  $G_j^a$  is either of the form  $(\pi_{a,j}(I_\nu), \pi_{a,j}(D_\nu))$ , for some  $\nu \in \mathcal{V}_K$ , or *cyclic*. Let  $\tilde{G}^c$  be a central extension as in Corollary 4.3, surjecting onto  $G^c$ . We have classes  $\tilde{\alpha}_j^a \in H^j(\tilde{G}^a)$ , induced by  $\alpha_j^a$  constructed above, and mapping to the same class  $\tilde{\alpha}^c \in H^j(\tilde{G}^c)$ .

We claim that  $\tilde{\alpha}^c \in H_{s,nr}^j(\tilde{G}^c)$ . Indeed, for every  $\Delta$ -pair  $(\tilde{I}, \tilde{D})$  in  $\tilde{G}^a$  either  $\tilde{D}$  projects to a cyclic group in  $G^a$  or it is extendable, i.e., image of some  $(I_\nu, D_\nu)$ . In the first case, all elements  $\tilde{\alpha}_j$  are unramified on  $(\tilde{I}, \tilde{D})$ . In the second case, at least one of the  $\tilde{\alpha}_j$  is unramified on it.

## 9. REDUCTION TO THE SMOOTH CASE

In absense of resolution of singularities in positive characteristic, we reduce to the smooth case via the de Jong-Gabber alterations theorem (see [17]): The Galois group  $G_K$  contains a subgroup  $G_{\tilde{K}}$  of finite index, coprime to  $\ell$ , such that

- The function field  $\tilde{K}$  corresponding to  $G_{\tilde{K}}$  admits a smooth model, i.e., there exists a finite cover

$$\rho : \tilde{X} \rightarrow X,$$

of degree  $|G_K/G_{\tilde{K}}|$  with  $\tilde{X}$  smooth and  $\tilde{K} = k(\tilde{X})$ .

Let  $\alpha_K \in H_{nr}^n(G_K)$  be an unramified class. Its restriction  $\alpha_{\tilde{K}}$  to a class in  $H^n(G_{\tilde{K}})$  is also unramified. By results in Section 8, there exists a surjection

$$(9.1) \quad G_{\tilde{K}} \rightarrow \tilde{G}^c$$

onto a finite abelian  $\ell$ -group such that  $\alpha_{\tilde{K}}$  is unduced from a class in  $\tilde{\alpha}^c \in H_{s,nr}^n(\tilde{G}^c)$ .

**Lemma 9.1.** *There exists a diagram*

$$\begin{array}{ccccc} G_{\tilde{K}} & \longrightarrow & \tilde{G} & \xrightarrow{\tilde{\pi}_c} & \tilde{G}^c \\ \downarrow & & \downarrow & & \\ G_K & \longrightarrow & G & & \end{array}$$

where the vertical arrows are injections, with image of index coprime to  $\ell$ ,  $\tilde{G}$  and  $G$  are finite groups, and  $\alpha_K$  is induced from an element  $\alpha_G \in H_{nr}^n(G)$ . In particular,  $\text{Syl}_{\ell}(G) \simeq \text{Syl}_{\ell}(\tilde{G})$ .

*Proof.* Fix a finite continuous quotient  $G_K \rightarrow G'$  such that  $\alpha_K$  is induced from some  $\alpha_{G'} \in H^i(G')$ . Note that for every intermediate quotient

$$G_K \rightarrow G \rightarrow G'$$

there exists an  $\alpha_G \in H^i(G)$  inducing  $\alpha_K$ . It suffices to find a sufficiently large  $G$  such that the surjection (9.1) factors through a subgroup of  $G$ . This is a standard fact in Galois theory.

Since  $\tilde{\alpha}^c \in H_{s,nr}^i(G^c)$  its image  $\tilde{\alpha} \in H^i(\tilde{G})$  is also unramified. Since the index  $(G : \tilde{G})$  is coprime to  $\ell$ , and since the unramified  $\tilde{\alpha}$  is induced from an element  $\alpha_G \in H^i(G)$ ,  $\alpha_G$  is also unramified, as claimed.  $\square$

At this stage, we cannot yet guarantee that  $G$  is a central extension of an abelian group, nor that it is an  $\ell$ -group. However, we know that

$$\text{tr}(\text{res}_{G_K/G_{\tilde{K}}}(\alpha_K)) = \alpha_K \in H_{nr}^i(G_K),$$

modulo multiplication by an element in  $(\mathbb{Z}/\ell^n)^\times$ .

We need the following version of resolution of singularities in positive characteristic:

**Theorem 9.2.** [18] *Let  $G$  be a finite  $\ell$ -group and  $Y$  a smooth variety with a generically free action of  $G$ . Then there exists a  $G$ -variety  $\tilde{Y}$  with a proper  $G$ -equivariant birational map  $\tilde{Y} \rightarrow Y$  such that  $\tilde{Y}/G$  is smooth.*

We are very grateful to D. Abramovich for providing the reference and indicating the main steps of the proof in [18]:

- Theorem VIII.1.1 gives an equivariant modification  $Y'$  of  $Y$  with a regular log structure on  $Y'$  such that the action is *very tame*, i.e., the stabilizers in  $G$  of points in  $Y'$  are abelian and act as subgroups of tori in toroidal charts of the log structure  $Y'$ .
- By Theorem VI.3.2, the quotient  $Y'/G$  of a log regular variety by a very tame action is log regular.
- By Theorem VIII.3.4.9, which is a step in Theorem VIII.1.1, a log regular variety has a resolution of singularities.

We return to the proof of Theorem 7.1. Start with a suitable faithful representation  $V$  of  $G$ , and thus of  $\mathrm{Syl}_\ell(G) = \mathrm{Syl}_\ell(\tilde{G})$ , and construct a diagram

$$\begin{array}{ccc} \bar{V} & \xrightarrow{\pi_Y} & Y = \tilde{Y}/\mathrm{Syl}_\ell(G) \\ \pi_G \downarrow & & \\ \bar{V}/G & & \end{array}$$

where  $Y = \tilde{Y}/\mathrm{Syl}_\ell(G)$  is the smooth projective variety from Theorem 9.2, and  $\pi_Y$  is a  $\mathrm{Syl}_\ell(G)$ -equivariant map from a  $G$ -equivariant projective closure of  $V$ . Let  $L = k(Y)$  be the function field of  $Y$ . Given a class  $\alpha_L \in H_{nr}^i(G_L)$  we have a covering of  $Y = \cup_n Y_n$  by affine Zariski open subsets and a finite set of classes  $\{\alpha_n\}$  representing  $\alpha_L$ , as considered in Section 8.

Pick a point  $v \in \bar{V}$ . The image  $S := \pi_Y(G \cdot v)$  of its  $G$ -orbit is a finite set of points. By Lemma 8.1, there exist a dense Zariski open subset  $U_S$  and a class  $\alpha_S \in H_{et}^i(U_S)$  coinciding with  $\alpha_L$  on some dense Zariski open subset  $U_S^\circ \subset U_S$ . Its preimage

$$\bar{U}_S^\circ := \pi_Y^{-1}(U_S^\circ) \subset \bar{V}$$

is a dense Zariski open subset containing  $G \cdot v$ . Put

$$\bar{U}_v := \cap_{g \in G} g(\bar{U}_S^\circ) \subset \bar{V},$$

it is a  $G$ -stable dense Zariski open subvariety containing  $v$ . Its image  $\pi_G(\bar{U}_v) \subset \bar{V}/G$  is a Zariski open subset containing  $\pi_G(G \cdot v)$ . Note that

$\pi_v : \bar{U}_v/\mathrm{Syl}_\ell(G) \rightarrow U_S$  is a birational morphism to an open subset and  $\pi_v^*(\alpha_S)$  is well-defined in étale cohomology of  $\bar{U}_v/\mathrm{Syl}_\ell(G)$ . It follows that the trace

$$\mathrm{tr}_{\pi_G}(\pi_v^*(\alpha_S)) \in H_{\mathrm{et}}^i(\bar{U}_v/G)$$

is well-defined and coincides with  $(G : \mathrm{Syl}_\ell(G)) \cdot \alpha_S$  at the generic point of  $\bar{V}/G$ .

Thus we have a covering of  $\bar{V}/G$  by Zariski open subsets of the form  $\bar{U}_v/G$ ,  $v \in \bar{V}$ , with cohomology classes representing  $\alpha_K$  on each chart. There exists a finite subcovering by  $\bar{U}_v/G$  with extensions of  $\alpha_S$  to each  $\bar{U}_v/G$ . We can now apply Proposition 5.2 to produce a finite subcover such that on each chart, the class is induced from homomorphisms onto finite *abelian* groups, and proceed as in Section 8.

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